Upper Bound on the Price of Anarchy in Content Placement Games for Caching

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Outline

1. Problem Description
2. Results
3. Q&A

Upper Bound on the **Price of Anarchy** in **Content Placement Games** for Caching

- Price of Anarchy
- Content Placement Game

Game Theory:

- Selfish and Unbounded Rational Players
  - Cost function for each player: $c_i(s)$.
  - s is a strategy profile consists of strategies of all players.
- Nash Equilibrium v.s. Optimal Strategy Profile.
  - Social Cost Function $c(s)$
  - No one can benefit from unilateral change.
  - Public Goods Tragedy.

What is Price of Anarchy?

$$PoA = \frac{c(s)}{c(s^*)}$$

where s is the worst Nash equilibrium strategy and s^* is the optimal strategy.

PoA measures the efficiency of a system degrades due to selfish behavior of its agents.

Content Placement Game

- A User
  - Appear in the whole region uniformly and at random. p_T : the probability of the user appear in region T.
- N Base Stations : N Players
  - Same capacity k.
- J Files $\{1, 2, ..., J\}$, w.l.o.g Assume $J = N \times k$
  - We use Zipf distribution to describe the probability of the user requesting for file j denoted as a_j.

$$a_j = \frac{j^{-\gamma}}{\sum_{i=1}^{J} i^{-\gamma}}$$

Cost Function for Each Player i

- The probability that the user can't find the file requested within the region covered by the player i.

$$c_i(s) = \sum_{j=1}^{J} a_j(1 - b_j^{(i)}) \sum_{M \in \Theta, i \in M} p_M \prod_{l \in M \setminus \{i\}} (1 - b_j^{(l)}) \quad (1)$$

  - where Θ is the power set of [1:N], M is a subset of base stations, s is an arbitrary strategy profile, $b_j^{(i)}$ is the indicator of the incidence that whether player i caches file j.

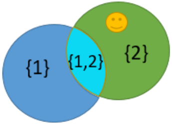
Social Cost Function

- The probability that the user can't find the requested file in the whole region.

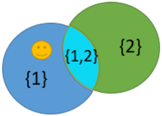
$$c(s) = \sum_{j=1}^{J} a_j \sum_{M \in \Theta} p_s \prod_{l \in M} (1 - b_j^{(l)}) \tag{2}$$

  - where Θ is the power set of [1:N], M is a subset of base stations, s is an arbitrary strategy profile, $b_j^{(i)}$ is the indicator of the incidence that whether player i caches file j.

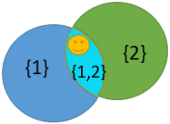
A Simple Example - Two Players, Three Files with Capacity One



(a)



(b)



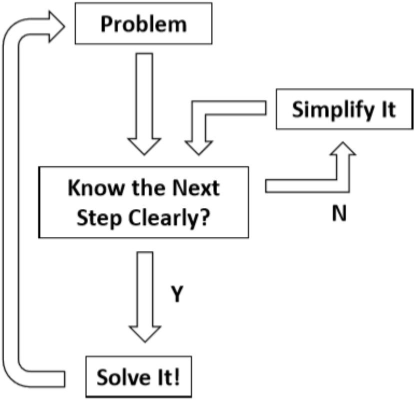
(c)

Assume player blue caches file 1, player green caches file 2, area of a disc is 1, the intersecting area is A. With this strategy profile.

$$c_{blue}(s) = a_2 \frac{1-A}{2-A} + a_3 \frac{1}{2-A},\ c_{green}(s) = a_1 \frac{1-A}{2-A} + a_3 \frac{1}{2-A}$$

$$c(s) = a_1 \frac{1-A}{2-A} + a_2 \frac{1-A}{2-A} + a_3$$

Start from Simple Case



General Case→Grid Case→Three Players→Two Players

Preliminary Results: Two Player Case

Theorem

In two-player case, assume the intersecting area is A, if $A < \frac{1}{2}$, then $\frac{1-A}{1-2A}$ is an upper bound on PoA.

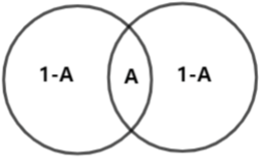
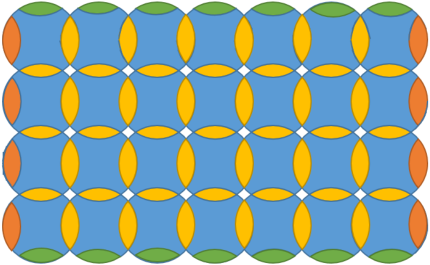


Figure: 2-player case

Improve this result: More players

Theorem

Assume base stations are fixed on a grid with $m \times n$ vertices, they intersect with their neighbors pairwise. A denotes the intersecting area. In this model, $\frac{4A}{1-8A}$ is an upper bound on PoA.



Try to Improve This Bound for More General A:

A Obvious Limitation about Theorems Above.

- When $A \to \frac{1}{2}, \frac{1}{8}$, the upper bound $\to \infty$.
- Where is this limitation from?

Methodology to get an Upper Bound on PoA in Theorems

The main idea of theorems is based on an inequality.

$$\lambda c(s^*) \geq \sum_{i=1}^{N} (c_i(s_i^*, s_{-i}) - c_i(s)) + (1-\mu)c(s) \qquad (3)$$

By bounding $\sum_{i=1}^{N} c_i(s_i^*, s_{-i})$ and $\sum_{i=1}^{N} c_i(s)$ with terms of $c(s^*)$, $c(s)$

- where s is the Nash equilibrium and s^* is the optimal solution.

Principle of The Inequality

Apply the definition of Nash equilibrium

$$\lambda c(s^*) \geq \sum_{i=1}^{N} (c_i(s_i^*, s_{-i}) - c_i(s)) + (1-\mu)c(s) \geq (1-\mu)c(s) \tag{4}$$

After moving terms, we get

$$\frac{c(s)}{c(s^*)} \leq \frac{\lambda}{1-\mu}$$

$\forall\ s \rightarrow$ Nash equilibrium and $s^* \rightarrow$ optimal strategy

Improving the Inequality

Bounding $\sum_{i=1}^N c_i(s_i^*, s_{-i})$ and $\sum_{i=1}^N c_i(s)$ with Terms of $c(s^*)$, $c(s)$...

- $$c(s) \leq \sum_{i=1}^{2} c_i(s) \leq \frac{2}{2-A} c(s)$$
- $$c(s) \leq \sum_{i=1}^{N} c_i(s)$$

We need a better understanding of the relationship between Nash equilibrium and optimal strategy.

Two-Player Case. Four Lemmas : Lemma 1

Lemma

(Rail Lemma) All the strategy profiles which are the candidates of Nash equilibrium and optimal strategy should in the form that:

- *First have successive same choices, then following complementary choices: one file is only cached by one player.*
- *No jumpings: if* $|\{i_1, i_2, ..., i_k\} \cup \{j_1, j_2, ..., j_k\}| = q$, *then*$\{i_1, i_2, ..., i_k\} \cup \{j_1, j_2, ..., j_k\} = \{1, ..., q\}$ *where* $k \leq q \leq 2k$

Illustrations

Figure: Not Successive Same Choices

Figure: Jumping

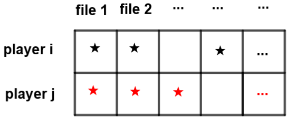


Figure: Qualified

Lemma 2-3

Lemma

With the basic settings, when $A \geq 1 - \frac{a_{2k}}{a_1}$ *(best bound), caching complementary files is an optimal and also a Nash equilibrium strategy and this is the only possible combination.*

Lemma

With basic settings, when $A \leq 1 - \frac{a_{k+1}}{a_k}$ *(best bound), caching* $\{1, 2, ..., k\} \times \{1, 2, ..., k\}$ *is a Nash equilibrium and also the optimal strategy, this is also the only possible combination.*

Lemma 4

Lemma

The completely same and completely complementary case strategy profile is impossible to "coexist": One is Nash equilibrium while the other is optimal, here we assume the capacity is strictly larger than 1.

- "Coexistence" worth further study.
- Method applied to obtain these four lemmas is not general enough.

$$Q\&A$$